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Influence functions, followers and command games

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Abstract. We study and compare two frameworks: a model of influence, and command games. In the influence model, in which players are to make a certain acceptance/rejection decision, due to influence of other players, the decision of a player may be different from his inclination. We study a relation between two central concepts of this model: influence function, and follower function. We deliver sufficient and necessary conditions for a function to be a follower function, and we describe the structure of the set of all influence functions that lead to a given follower function. In the command structure introduced by Hu and Shapley, for each player a simple game called the command game is built. One of the central concepts of this model is the concept of command function. We deliver sufficient and necessary conditions for a function to be a command function, and describe the minimal sets generating a normal command game. We also study the relation between command games and influence functions. A sufficient and necessary condition for the equivalence between an influence function and a normal command game is delivered.

JEL Classification: C7, D7

Keywords: influence function, follower function, lower and upper inverses, kernel, command game, command function, minimal sets generating a command game

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1 Introduction

One of the natural phenomena that may appear in voting situations, and consequently should be studied in detail, is *influence* or *interaction* among players. Among concepts related to this topic it is worth mentioning the notion of *influence relation* in simple games, which was introduced fifty years ago in [19], to qualitatively compare the a priori influence of voters in a simple game. As defined in [19], in a simple game, where players can vote either YES or NO, voter k is said to be at least as influential as voter j, if whenever j can transform a loosing coalition into a majority by joining it, voter k can achieve the same *ceteris paribus*. Very recently, in [24] the influence relation has been extended to voting games with abstention. The concept of interaction among players in a cooperative game is also studied, for instance, in [12], where players in a coalition are said to exhibit a positive (negative) interaction when the worth of the coalition is greater (smaller) than the sum of the individual worths.

Another approach to modeling players' interactions has been proposed in [15], where influence among players in a social network is analyzed. In this model, each player is assumed to have an *inclination* to say either YES or NO which, due to influence by other voters, may be different from the *decision* of the player. The influence model introduced in [15] differs from other voting models, because in the influence model the analysis begins not in voting itself, but 'one step earlier', that is, in the framework of original inclinations of voters. *Influence* by other players in this model means that a player's vote (decision) is different from his original inclination. Formally, the influence is expressed by an *influence function*, which assigns to each inclination vector (i.e., a vector describing the inclinations of all players) a decision vector (i.e., a vector indicating the decisions of the players). One of the tools that describe the influence function is the concept of a follower of a given coalition under a given influence function, that is, a voter who always follows the inclination of the coalition in question. Formally, a *follower function*, which assigns to each coalition the set of its followers, is defined. This influence model is studied in [14] where, in particular, we introduce weighted influence indices, and consider different influence functions.

Another interesting model related to the topic in question has been recently presented in [17, 18], where the command structure of Shapley [23] is applied to model players' interaction relations by simple games. For each player, boss sets and approval sets are introduced, and based on these sets, a simple game called the *command game* for a player is built. Given a set of command games, the *command function* is defined, which assigns to each coalition the set of all players that are 'commandable' by that coalition. In [13] we compare the framework of command games with the influence model. In particular, we define several influence functions which capture the command structure. These functions are compatible with the command games, in the sense that each commandable player for a coalition in the command game is a follower of the coalition under the command influence function. Some of the presented influence functions are equivalent to the command games: An influence function and a command game are said to be *equivalent* if the follower function of this influence function is identical to the command function in this command game. For some influence functions we define the equivalent command games. Moreover, we show that not for all influence functions the compatible command games exist.

The aim of the present paper is to continue our work presented in [13, 14]. While in these two previous papers on influence we focus on concrete examples, for instance, we define several influence functions and study their properties, the aim of the present paper is to establish *exact relations between the key concepts of the influence model and the framework of command games*. To be more precise, the aims and main results of this paper are the following:

- *studying the exact relation between an influence function and a follower function* - We deliver sufficient and necessary conditions for a function to be the follower function of some influence function. Given a follower function, we find the smallest and greatest influence functions, called the *lower and upper inverses*, that lead to this follower function. Moreover, we describe the structure of the set of all influence functions that lead to a given follower function. This structure happens to be a distributive lattice, and we indicate how to compute it.
- *studying the exact relation between a command game and a command function* - We deliver sufficient and necessary conditions for a function to be the command function of some command game. Moreover, we describe the *minimal sets (winning coalitions) generating a normal command game*.

– *studying the exact relation between a command game and an influence function* - A sufficient and necessary condition for the equivalence between an influence function and a normal command game is delivered. We calculate the *kernel* of an influence function equivalent to a normal command game. Several examples that illustrate the concepts studied and results obtained in this paper are presented.

The paper is structured as follows. In Section 2 we present basic notations and definitions, related to partially ordered sets. Section 3 concerns the model of influence. We recapitulate briefly the model, and study the relation between influence functions and followers functions. In Section 4, the relation between a command game and a command function, and the relation between a command game and an influence function, are studied. In Section 5, we give some concluding remarks, enhancing the most important results of the paper. Long and technical proofs, as well as the technical material on partially ordered sets and lattices, are put in the appendix.

2 Some notations and definitions

We give here some essential definitions and notations used in the paper, which are borrowed from the field of partially ordered sets. More will be given in the appendix, essentially concerning lattices.

We begin by giving some conventions for sets. We often omit braces for sets if no confusion occurs, e.g., $N\setminus\{k\}$, $S\cup\{k\}$ will be written $N\setminus k$, $S\cup k$, etc. Set complementation will be often denoted by a bar, i.e., $\overline{S} := N \setminus S$, where N is the referential set, and S a subset of it.

Given a finite set N, we often deal in this paper with functions from 2^N to 2^N. Similarly as we write for real-valued functions $f \leq g$ for $f(x) \leq g(x)$ for all x, we write for two functions $F, G : 2^N \to 2^N$ that $F \leq G$ if $F(S) \subseteq G(S)$ for all $S \subseteq N$.

If neither $F \leq G$ nor $G \leq F$ hold, F and G are said to be *incomparable*. As usual, $F = G$ means $F(S) = G(S)$ for all $S \subseteq N$.

$F : 2^N \to 2^N$ is *isotone* or *monotone nondecreasing* if $S \subseteq T \subseteq N$ implies $F(S) \subseteq F(T)$. If the first inclusion is reversed, then F is said to be *antitone* or *monotone nonincreasing*. A function is *monotone* if it is either isotone or antitone.

A *partially ordered set* $(P, \leq)$ or *poset* for short, is a set P endowed with a partial order $\leq$, that is, a binary relation being reflexive, antisymmetric and transitive. $(2^N, \subseteq)$ is an example of poset, as well as the set of functions from $2^N \to 2^N$, endowed with the above defined order. Generalizing the notion of interval, for two elements $x, y \in P$ such that $x \leq y$, we write $[x, y] := \{z \mid x \leq z \leq y\}$. Also we write $]x, y]$ if x is excluded from the interval, similarly for $[x, y[$. This notation will be often used in the sequel, for subsets and functions.

Given $x \in P$, a *predecessor* of x is any element y such that $y \leq x$.

A family of subsets is an upset if any superset of an element of the family belongs also to that family. For any $S \subseteq N$, we define $\uparrow S := \{T \subseteq N \mid T \supseteq S\}$, the *principal filter* of S. Evidently, for N being finite, any upset is a union of principal filters. The definition can be generalized to any poset.

3 Influence functions and follower functions

3.1 The model of influence

The framework of influence that we study in this paper has been originally introduced in [15], and next analyzed in [13, 14, 20–22]. We consider a social network with the set of players (agents, voters) denoted by $N := \{1, ..., n\}$. The players have to make a certain acceptance/rejection decision. Each player has an inclination either to say YES (denoted by $+1$) or NO (denoted by -1). An *inclination vector* $i = (i_1, ..., i_n)$ is an n-vector consisting of ones and minus ones, and indicating the inclinations of all players. Let $I := \{-1, +1\}^n$ be the set of all inclination vectors, and for any $S \subseteq N$, $|S| \geq 2$, let I_S denote the set of all inclination vectors under which all members of S have the same inclination, i.e.,

$$I_S := \{i \in I \mid \forall k, j \in S,\ i_k = i_j\}.$$

For convenience, we denote $(1, 1, \ldots, 1) \in I$ by 1_N, similarly for -1_N, and also for mixed cases like $(1_S, -1_{N\setminus S})$. This last notation suggests to use the more compact notation S, i.e., the set of YES voters, to denote the inclination vector $(1_S, -1_{N\setminus S})$. This set notation will be used most often in the paper.

It is assumed that players may influence each other, and due to the influences in the network, the (final) decision of a player may be different from his (original) inclination. In other words, each inclination vector $i \in I$ is transformed into a *decision vector* Bi, where $B : I \to I, i \mapsto Bi$ is the *influence function*. The decision vector $Bi = ((Bi)_1, ..., (Bi)_n)$ is an n-vector consisting of ones and minus ones, and indicating the decisions made by all players. The set of all influence functions is denoted by $\mathcal{B}$.

Using the set notation, if i corresponds to S, we denote Bi by $B(S)$, and $B(S) \subseteq N$ is the set of voters whose (final) decision is YES. Hence, an influence function can also be seen as a mapping from 2^N to 2^N.

One of the main concepts of the influence model is the concept of a *follower* of a given coalition, that is, a voter who 'always' follows the inclination of the coalition in question. 'Always' means here in all cases in which all members of the coalition have the same inclination. Let $B \in \mathcal{B}$. The *follower function* of B is a mapping $F_B : 2^N \to 2^N$ defined as

$$F_B(S) := \{k \in N \mid \forall i \in I_S,\ (Bi)_k = i_S\}, \quad \forall S \subseteq N, S \neq \emptyset,$$

and $F_B(\emptyset) := \emptyset$. $F_B(S)$ is the *set of followers of* S *under* B. In [14] it is shown that F_B is isotone, and $F_B(S) \cap F_B(T) = \emptyset$ whenever $S \cap T = \emptyset$. The set of all follower functions is denoted by $\mathcal{F}$.

In set notation, the definition of the follower function becomes:

$$F_B(S) = \bigcap_{S' \supseteq S} B(S') \cap \bigcap_{S' \subseteq N\setminus S} \overline{B(S')}, \quad \forall S \subseteq N, S \neq \emptyset, \tag{1}$$

and $F_B(\emptyset) := \emptyset$, as it can be checked.

Assume F_B is not identically the empty set. The *kernel* of B is the following collection of sets:

$$\mathcal{K}(B) := \{S \in 2^N \mid F_B(S) \neq \emptyset, \text{ and } S' \subset S \Rightarrow F_B(S') = \emptyset\}.$$

The kernel is well defined due to isotonicity of F_B. It is the set of minimal coalitions having followers, or put otherwise, the set of 'truly' influential coalitions.

3.2 The mapping Φ

We want to establish the exact relation between two key concepts of the influence model: the influence function, and the follower function. We have seen that an influence function B can be considered as a mapping from 2^N to 2^N, exactly like follower functions. The cardinality of the set of such mappings is $(2^n)^{(2^n)} = 2^{n2^n}$, and there are potentially as many influence functions as follower functions. However, while there is no restriction on B, F_B should satisfy some conditions, like isotonicity. Hence, there are functions in $(2^N)^{(2^N)}$ which cannot be the follower function of some influence function, and consequently, several B's may have the same follower function (put differently, we loose some information by considering only F_B). Formally, this means that the mapping $\Phi : \mathcal{B} \to (2^N)^{(2^N)}$, defined by

$$B \mapsto \Phi(B) := F_B$$

is neither a surjection nor an injection. We have $\Phi(\mathcal{B}) =: \mathcal{F}$.
The following natural questions may be raised:

(1) Given a function $F : 2^N \to 2^N$, which are the sufficient and necessary conditions so that there exists $B \in \mathcal{B}$ such that F is the follower function of B, i.e., $F = F_B$?
(2) If $F : 2^N \to 2^N$ is indeed a follower function, can we easily find examples of B's such that $F_B = F$?
(3) Moreover, can we find $\Phi^{-1}(F)$, i.e., the set of all influence functions that lead to the follower function F? What is the (algebraic) structure of $\Phi^{-1}(F)$?

The results shown in this subsection answers the first and second questions, and Subsection 3.3 deals with the third question.

Proposition 1. *A function $F : 2^N \to 2^N$ is a follower function of some $B \in \mathcal{B}$ (i.e., $F_B = F$, or $\Phi(B) = F$) if and only if it satisfies the following three conditions:*

(i) $F(\emptyset) = \emptyset$;
(ii) *F is isotone;*
(iii) *If $S \cap T = \emptyset$, then $F(S) \cap F(T) = \emptyset$.*

Moreover, the smallest and greatest influence functions belonging to $\Phi^{-1}(F)$ are respectively the influence functions $\underline{B}_F$ and $\overline{B}_F$, defined by, for all $i \in I$ and all $k \in N$:

$$(\underline{B}_F i)_k := \begin{cases} +1, & \text{if } k \in F(S^+(i)) \\ -1, & \text{otherwise} \end{cases},$$

$$(\overline{B}_F i)_k := \begin{cases} -1, & \text{if } k \in F(S^-(i)) \\ +1, & \text{otherwise} \end{cases},$$

where we denote for convenience $S^{\pm}(i) := \{j \in N \mid i_j = \pm 1\}$. We call these influence functions the lower and upper inverses *of F.*

Proof: We already know from [14, Prop. 2] that any follower function fulfills the above three conditions. Take $F : 2^N \to 2^N$ satisfying the above conditions. Let us check if indeed $\Phi(\underline{B}_F) =: F_{\underline{B}_F} = F$. We have to prove that $F_{\underline{B}_F}(S) = F(S)$ for all $S \subseteq N$. It is true for $S = \emptyset$, by definition of follower functions, and the condition $F(\emptyset) = \emptyset$.

We consider some subset $S \neq \emptyset$. Let us first study the case where $F(S) = \emptyset$. This implies that $\underline{B}_F i = (-1, \ldots, -1)$, for $i = (1_S, -1_{N\setminus S})$, which in turn implies that $F_{\underline{B}_F}(S) = \emptyset$.

Suppose now $F(S) \neq \emptyset$, and $k \in F(S)$. Let us show that $k \in F_{\underline{B}_F}(S)$. For $i = (1_S, -1_{N\setminus S}) \in I_S$, we have $(\underline{B}_F i)_k = 1 = i_S$. We have to show that this remains true for any $i \in I_S$. We have $I_S = \{(1_{S'}, -1_{N\setminus S'}) \mid S' \supseteq S\} \cup \{(-1_{S'}, 1_{N\setminus S'}) \mid S' \supseteq S\}$. If $i = (1_{S'}, -1_{N\setminus S'})$ for $S' \supseteq S$, we have $(\underline{B}_F i)_k = i_S = 1$ if $k \in F(S')$, which is true since $k \in F(S)$ and F is isotone. If $i = (-1_{S'}, 1_{N\setminus S'})$ for $S' \supseteq S$, we have $(\underline{B}_F i)_k = i_S = -1$ if $k \notin F(N\setminus S')$. Since $S\cap(N\setminus S') = \emptyset$, by the third condition we have $F(S)\cap F(N\setminus S') = \emptyset$, hence $k \notin F(N \setminus S')$. In conclusion, $k \in F_{\underline{B}_F}(S)$.

Conversely, if $F_{\underline{B}_F}(S) = \emptyset$, then necessarily $F(S) = \emptyset$ too, since we have proved above that any element in $F(S)$ is also in $F_{\underline{B}_F}(S)$. Suppose now that $F_{\underline{B}_F}(S) \neq \emptyset$, and take $k \in F_{\underline{B}_F}(S)$. Then for any $i \in I_S$, $(\underline{B}_F i)_k = i_S$. In particular, $i := (1_S, -1_{N\setminus S}) \in I_S$, so that $(\underline{B}_F i)_k = 1$, which implies that $k \in F(S)$.

Finally, $\underline{B}_F$ is the smallest B such that $\Phi(B) = F$ because any B in $\Phi^{-1}(F)$ must satisfy for any $k \in F(S) \neq \emptyset$, $Bi_k = 1$ for $i = (1_S, -1_{N\setminus S})$. Hence $B \geq \underline{B}_F$.

The proof for the upper inverse is analogous. ■

Using the set notation, the above results can be written in a much simpler way. It is easy to see that, for all $S \subseteq N$,

$$\underline{B}_F(S) = F(S), \quad \overline{B}_F(S) = \overline{F(\overline{S})}. \tag{2}$$

Example 1. Consider $F(S) = \emptyset$, for all $S \subseteq N$, which is a follower function. We already know from [14, Prop. 7] that an inverse of F by Φ is the reversal function $-\mathsf{Id}$, defined by $(-\mathsf{Id})i := -i$, for each $i \in I$. Clearly, the lower inverse is the constant function $B \equiv -1_N$, while the upper inverse is $B \equiv 1_N$.

Example 2. Consider $F = \mathsf{Id}$, which is a follower function. We know already from [14, Prop. 6] that an inverse of F is the identity function Id. Clearly, the lower and upper inverses collapse to Id. Hence, $\Phi^{-1}(\mathsf{Id}) = \{\mathsf{Id}\}$.

Example 3. Let $n = 3$ and the following function F be defined as follows:

S	$\emptyset$	1	2	3	12	13	23	123
$F(S)$	$\emptyset$	$\emptyset$	2	$\emptyset$	2	3	12	123

It can be checked that it is indeed a follower function. Then the upper and lower inverses are, using set notation:

i	$\emptyset$	1	2	3	12	13	23	123
$\overline{B}_F i$	$\emptyset$	3	12	13	123	13	123	123
$\underline{B}_F i$	$\emptyset$	$\emptyset$	2	$\emptyset$	2	3	12	123

The following proposition is easily deduced from (2).

Proposition 2. *The function Φ satisfies the following properties:*

(i) For any $B \in \mathcal{B}$, $\Phi(B) \leq B$.
(ii) The set of fixed points of Φ (i.e., for which $\Phi(B) = B$) is exactly $\mathcal{F}$. Hence, $\Phi^2 = \Phi^3 = \cdots = \Phi$.

Proof: (i) Denoting $F := \Phi(B)$, by definition $B \geq \underline{B}_F = F = \Phi(B)$ by (2).

(ii) If $B \in \mathcal{F}$, then $B \in \Phi^{-1}(B)$, hence $\Phi(B) = B$. Conversely, suppose that $B \notin \mathcal{F}$ and $\Phi(B) = B$. But then $B \in \mathcal{F}$, a contradiction. The last affirmation follows from $\Phi(\mathcal{B}) = \mathcal{F}$. ■

Remark 1. The function Φ is not monotone (neither isotone nor antitone), so it fails to be a dual closure operator (see Appendix for definition). The following example shows this fact.

S	$\emptyset$	1	2	3	12	13	23	123
$B(S)$	$\emptyset$	$\emptyset$	2	$\emptyset$	2	3	12	123
$\Phi(B)(S)$	$\emptyset$	$\emptyset$	2	$\emptyset$	2	3	12	123
$B'(S)$	$\emptyset$	$\emptyset$	12	2	12	3	12	123
$\Phi(B')(S)$	$\emptyset$	$\emptyset$	1	$\emptyset$	1	3	12	123

Clearly, $B' \geq B$, but $\Phi(B)$ and $\Phi(B')$ are incomparable.

3.3 Structure of $\Phi^{-1}(F)$

N.B. All definitions concerning lattices are put in the Appendix.

We know that all elements of the inverse of F are between $\underline{B}_F$ and $\overline{B}_F$, with the usual order $\leq$ on functions. Then $(\Phi^{-1}(F), \leq)$ is a poset, which is a subset of $([\underline{B}_F, \overline{B}_F], \leq)$. We write for simplicity

$$D_S := \overline{B}_F(S) \setminus \underline{B}_F(S), \quad S \subseteq N.$$

Hence, an element of $[\underline{B}_F, \overline{B}_F]$ is more easily denoted by the 2^n-dim vector $(T_\emptyset, \dots, T_N)$, where $T_S \subseteq D_S$ for each $S \subseteq N$. With this notation, $\underline{B}_F$ and $\overline{B}_F$ are denoted by $(\emptyset, \dots, \emptyset)$ and $(D_\emptyset, \dots, D_N)$ respectively, and $\overline{B}_F$ in Example 3 is $(\emptyset, 3, 1, 13, 13, 1, 3, \emptyset)$. Moreover, $[\underline{B}_F, \overline{B}_F]$ is simply $\prod_{S \subseteq N} 2^{D_S}$, hence it is a Boolean lattice.

We begin by a simple but fundamental observation.

Remark 2. Let $F \in \mathcal{F}$. For any $S \subseteq N$, we have $D_S = D_{\overline{S}}$. Indeed, since $F(S) \cap F(\overline{S}) = \emptyset$ by Proposition 1 (iii),

$$D_S = \overline{B}(S) \setminus \underline{B}(S) = \overline{F(\overline{S})} \setminus F(S) = \overline{F(S)} \setminus F(\overline{S}) = \overline{B}(\overline{S}) \setminus \underline{B}(\overline{S}) = D_{\overline{S}}.$$

Due to this, T_S and $T_{\overline{S}}$ neither intersect $F(S)$ nor $F(\overline{S})$ (see Figure 1).

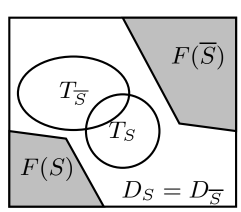

Fig. 1. Set relations between $F(S)$ and T_S

A necessary and sufficient condition for a given B to belong to $\Phi^{-1}(F)$ is given in the next proposition.

Proposition 3. *Let $F \in \mathcal{F}$ be given, and consider $B := (T_\emptyset, \dots, T_N)$ an element of $[\underline{B}_F, \overline{B}_F]$. Then $B \in \Phi^{-1}(F)$ if and only if the following conditions are satisfied:*

(i) $\bigcap_{S \ni i} T_S \setminus T_{\overline{S}} = \emptyset$, $\forall i \in N$.

(ii) $\bigcap_{S' \in \mathcal{C}} F(S') \cap \bigcap_{\substack{S'' \supseteq S \\ S'' \not\supseteq S', \forall S' \in \mathcal{C}}} T_{S''} \setminus T_{\overline{S''}} = \emptyset$, *for any antichain $\mathcal{C}$ in $]S, N]$ such that* $\bigcap_{S' \in \mathcal{C}} F(S') \neq F(S)$, *for any $S \subset N$.*

(see proof in Appendix)

Remark 3. (i) is equivalent to $\bigcap_{S' \supseteq S} T_{S'} \setminus T_{\overline{S'}} = \emptyset$, $\forall S \subseteq N$, $S \neq \emptyset$. In the proof, the latter is shown. In (ii), the condition $\bigcap_{S' \in \mathcal{C}} F(S') \neq F(S)$ can be removed. It serves only to reduce the complexity when computing this condition.

Conditions (i) and (ii) are somewhat complicated, but necessary to prove the main result of this section (Theorem 1). A much less general result, but much simpler and still useful, is given in the next proposition. We begin by a technical lemma.

Lemma 1. *Let B, B' be identical influence functions except on $S \subseteq N$, where $k \not\in B(S)$, and $B'(S) = B(S) \cup k$. Assume that k is neither a follower of S in B' nor a follower of $\overline{S}$ in B (i.e., $k \not\in F_{B'}(S)$, $k \not\in F_B(\overline{S})$). Then $F_B(S') = F_{B'}(S')$, for all $S' \subseteq N$.*

(see proof in Appendix)

Proposition 4. *Let $B := (T_\emptyset, \dots, T_N) \neq \overline{B}_F$ be an element of $\Phi^{-1}(F)$. Then for any $S \subseteq N$ such that $D_S \setminus T_S \neq \emptyset$, and any $k \in D_S \setminus T_S$, $B' := (T_\emptyset, \dots, T_S \cup \{k\}, \dots, T_N)$ is an element of $\Phi^{-1}(F)$ if and only if one of the following conditions is not satisfied:*

(i) For any $S' \supset S$, $k \in B(S')$
(ii) For any $S' \subseteq N \setminus S$, $k \not\in B(S')$.

Proof: Before to show the equivalence, we remark that $k \in D_S \setminus T_S$ implies that $k \not\in F(S)$ and $k \not\in F(\overline{S})$ (by definition of $\overline{B}_F$, see Figure 1). Remember also that $F = F_B$.

Assume that the two conditions hold. Then k is a follower of S for B', which implies that $B' \not\in \Phi^{-1}(F)$.

Assume on the contrary that one of the conditions is false. We have to prove that $F_{B'}(S') = F_B(S')$, for all $S' \subseteq N$. Consider first the case $S' = S$. Since the only change concerns k, changes of the followers for B and B' can only concern k. Then we have only one possibility: k is not a follower of S for B but it becomes for B'. Since one of the conditions fails, $k \in B'(S)$ is not sufficient to ensure that $k \in F_{B'}(S)$. Hence $k \not\in F_{B'}(S)$, and since $k \not\in F_B(\overline{S})$, we are exactly in the conditions of Lemma 1, which proves the equality of F_B and $F_{B'}$. ■

We describe now the structure of $\Phi^{-1}(F)$.

Theorem 1. *For any $F \in \mathcal{F}$, the set $\Phi^{-1}(F)$, endowed with the usual ordering of functions, has the following properties:*

(i) *Top and bottom elements are $\overline{B}_F$ and $\underline{B}_F = F$.*
(ii) *It is a lattice, with supremum and infimum given by*

$$(B \vee B')(S) := B(S) \cup B'(S)$$
$$(B \wedge B')(S) := B(S) \cap B'(S)$$

for any $S \in 2^N$, and hence it is a sublattice of the product lattice $\prod_{S \subseteq N} 2^{D_S}$.

(iii) *$\Phi^{-1}(F)$ is autodual, i.e., $(\Phi^{-1}(F), \leq)$ and $(\Phi^{-1}(F), \geq)$ are isomorphic. The duality is expressed as follows: to each element $B := (T_\emptyset, \dots, T_S, \dots, T_N)$ of $\Phi^{-1}(F)$ corresponds the element $B' := (D_N \setminus T_N, \dots, D_{\overline{S}} \setminus T_{\overline{S}}, \dots, D_\emptyset \setminus T_\emptyset)$.*
(iv) *There are $\sum_{S \subseteq N} |D_S|$ join-irreducible elements, one for each $k \in D_S$, $S \subseteq N$, either of the form $(k_S\emptyset)$ if this element belongs to $\Phi^{-1}(F)$, otherwise of the form $(k_S k_{\overline{S}}\emptyset)$, where the notation $(k_S\emptyset)$ is a shorthand for $(\emptyset, \dots, \emptyset, k, \emptyset, \dots, \emptyset)$, where k is at position S, and similarly for $(k_S k_{\overline{S}}\emptyset)$.*
(v) *The lattice is distributive, hence ranked, and its height is $h = \sum_{S \subseteq N} |D_S|$.*

(see proof in Appendix) Concerning (iv), checking whether $(k_S\emptyset)$ is an element of $\Phi^{-1}(F)$ is done by using Proposition 4 with $(T_\emptyset, \dots, T_N) = (\emptyset, \dots, \emptyset)$, i.e., it amounts to check if one of the two following conditions fails:

(i) $\forall S' \supset S$, $k \in F(S')$
(ii) $\forall S' \subseteq N \setminus S$, $k \notin F(S')$.

Note that if $(k_S\emptyset)$ is not an element of $\Phi^{-1}(F)$, then necessarily $(k_{\overline{S}}\emptyset)$ is.

As explained in the Appendix, from standard results of lattice theory, the sole knowledge of the join-irreducible elements permits to build the entire lattice when it is distributive. Consequently, the above theorem describes $\Phi^{-1}(F)$ entirely, and allows its practical computation.

Example 4. (Example 3 continued) Let us compute the join-irreducible elements of $\Phi^{-1}(F)$, with F given in Example 3. The sets D_S are as follows.

S	$\emptyset$	1	2	3	12	13	23	123
D_S	$\emptyset$	3	1	13	13	1	3	$\emptyset$

We have:

- For $S = 1, k = 3$: $(3_1\emptyset)$ belongs to $\Phi^{-1}(F)$, so it is a join-irreducible element.
- For $S = 2, k = 1$: $(1_2\emptyset)$ belongs to $\Phi^{-1}(F)$, so it is a join-irreducible element.
- For $S = 3, k = 1$: $(1_3\emptyset)$ belongs to $\Phi^{-1}(F)$, so it is a join-irreducible element.
- For $S = 3, k = 3$: $(3_3\emptyset)$ belongs to $\Phi^{-1}(F)$, so it is a join-irreducible element.
- For $S = 12, k = 1$: $(1_{12}\emptyset)$ does not belong to $\Phi^{-1}(F)$, so $(1_{12}1_3\emptyset)$ is a join-irreducible element.
- For $S = 12, k = 3$: $(3_{12}\emptyset)$ does not belong to $\Phi^{-1}(F)$, so $(3_{12}3_3\emptyset)$ is a join-irreducible element.
- For $S = 13, k = 1$: $(1_{13}\emptyset)$ does not belong to $\Phi^{-1}(F)$, so $(1_{13}1_2\emptyset)$ is a join-irreducible element.

– For $S = 23, k = 3$: $(3_{23}\emptyset)$ does not belong to $\Phi^{-1}(F)$, so $(3_{23}3_1\emptyset)$ is a join-irreducible element.

The next proposition is the dual of Proposition 4, and follows directly from the dual structure of $\Phi^{-1}(F)$.

Proposition 5. *Let $B := (T_\emptyset, \dots, T_N) \neq \underline{B}_F$ be an element of $\Phi^{-1}(F)$. Then for any $S \subseteq N$ such that $T_S \neq \emptyset$, and any $k \in T_S$, $B' := (T_\emptyset, \dots, T_S \setminus \{k\}, \dots, T_N)$ is an element of $\Phi^{-1}(F)$ if and only if one of the following conditions is not satisfied:*

(i) For any $S' \supset \overline{S}$, $k \in \overline{B}(S')$
(ii) For any $S' \subseteq S$, $k \notin \overline{B}(S')$.

with $\overline{B} := (D_N \setminus T_N, \dots, D_{\overline{S}} \setminus T_{\overline{S}}, \dots, D_\emptyset \setminus T_\emptyset)$.

Proof: Assuming $B = (T_\emptyset, \dots, T_S, \dots, T_{\overline{S}}, \dots, T_N) \in \Phi^{-1}(F)$ gives that $\overline{B} := (D_N \setminus T_N, \dots, D_{\overline{S}} \setminus T_{\overline{S}}, \dots, D_S \setminus T_S, \dots, D_\emptyset \setminus T_\emptyset)$ is also in $\Phi^{-1}(F)$ by Th. 1 (iii). Now, $B' = (T_\emptyset, \dots, T_S \setminus k, \dots, T_{\overline{S}}, \dots, T_N) \in \Phi^{-1}(F)$ if and only if $\overline{B'} = (D_N \setminus T_N, \dots, D_{\overline{S}} \setminus T_{\overline{S}}, \dots, (D_S \setminus T_S) \cup k, \dots, D_\emptyset \setminus T_\emptyset)$ belongs to $\Phi^{-1}(F)$. We use then Proposition 4. ■

4 Command games, command functions, and influence functions

4.1 The command games

We recapitulate briefly the main concepts of the command games introduced by Hu and Shapley [17, 18]. Let $N = \{1, ..., n\}$ be the set of players (voters). For $k \in N$ and $S \subseteq N \setminus k$:

- S is a *boss set* for k if S determines the choice of k;
- S is an *approval set* for k if k can act with an approval of S.

It is assumed that no subset can be both a boss set and an approval set, and that any superset (in $N \setminus k$) of a boss set (resp. of an approval set) is a boss set (resp. an approval set, provided it is not a boss set). Also, it is assumed that the empty set cannot be a boss set because this does not make sense, but the empty set can be an approval set (which means that player k can act alone). To avoid triviality, it is assumed that both families of boss sets and approval sets cannot be empty.

For each $k \in N$, a simple game $(N, \mathcal{W}_k)$ is built, called the *command game for k*, where the set of winning coalitions is

$$\mathcal{W}_k := \{S \mid S \text{ is a boss set for } k\} \cup \{S \cup k \mid S \text{ is a boss or approval set for } k\}.$$

Note that due to the above assumptions, this family is never empty and always contains N. We call for brevity *command game* the set $\{(N, \mathcal{W}_k), k \in N\}$ of command games for each player.

We can recover the boss sets for k by

$$\mathsf{Boss}_k = \{S \subseteq N \setminus k \mid S \in \mathcal{W}_k\} = \mathcal{W}_k \cap 2^{N \setminus k},$$

and the approval sets for k by

$$\mathsf{App}_k = \{S \subseteq N \setminus k \mid S \cup k \in \mathcal{W}_k \text{ but } S \notin \mathcal{W}_k\}.$$

Moreover, we consider the *minimal boss sets* and the *minimal approval sets* for k

$$\mathsf{Boss}_k^* := \{S \in \mathsf{Boss}_k \mid S' \subset S \Rightarrow S' \notin \mathsf{Boss}_k\}$$

$$\mathsf{App}_k^* := \{S \in \mathsf{App}_k \mid S' \subset S \Rightarrow S' \notin \mathsf{App}_k\}.$$

Given a set of command games $\{(N, \mathcal{W}_k), k \in N\}$, the *command function* $\omega : 2^N \to 2^N$ is defined as

$$\omega(S) := \{k \in N \mid S \in \mathcal{W}_k\}, \ \forall S \subseteq N. \tag{3}$$

$\omega(S)$ is the *set of all members that are 'commandable' by* S. In [17] it is shown that $\omega(\emptyset) = \emptyset$, $\omega(N) = N$, and $\omega(S) \subseteq \omega(S')$ whenever $S \subset S'$.

4.2 Relation between command games and command functions

In Section 3, we have investigated the relation between influence functions and follower functions. Similar questions can be raised concerning command games and command functions, as well as the exact relationship between command games and influence functions. We start by studying the relation between command games and command functions.

We begin by a simple observation, and try to restrict the above framework, avoiding nonrealistic situations. This will lead to the notion of *normal command game*. A command game $\{(N, \mathcal{W}_k), k \in N\}$ can be viewed more compactly as a mapping $\Omega : N \times 2^N \to \{0, 1\}$, with

$$(k, S) \mapsto \Omega(k, S) = \begin{cases} 1, & \text{if } S \in \mathcal{W}_k \\ 0, & \text{otherwise} \end{cases}.$$

The set of such functions is $2^{N \times 2^N}$, hence its cardinality is $2^{n 2^n}$, which is exactly the cardinality of $\mathcal{B}$. However, not every such function corresponds to a command game. Indeed, if we examine the structure induced by boss and approval sets, we find that $\mathcal{W}_k$ is a union of principal filters, and hence it is an upset (see Section 2 for definitions):

$$\mathcal{W}_k = \uparrow S_1 \cup \ldots \cup \uparrow S_l$$

produced either by S_j not containing k (these S_j are minimal boss sets for k), or by S_j containing k (in this case, $S_j \setminus k$ are minimal approval sets for k).

We propose to impose in addition that $S_1 \cap \cdots \cap S_l \neq \emptyset$ for each $\mathcal{W}_k$. This implies in particular that there are no two disjoint boss sets. Indeed, if this condition is not true, then there exist for player k two disjoint coalitions which are winning. Since they are disjoint, it may be the case that one coalition votes YES and the other one votes NO, which would lead to a conflict.

Lastly, recall that the empty set cannot be a boss set, hence $\mathcal{W}_k \neq \uparrow \emptyset = 2^N$. This leads to the following definition.

Definition 1. *A* normal command game Ω *is a set of simple games* $\{(N, \mathcal{W}_k), k \in N\}$ *satisfying the two conditions:*

(i) For each $k \in N$, there exists a minimal nonempty family of nonempty subsets $S_1^k, \dots, S_{l_k}^k$ (called the generating family *of $\mathcal{W}_k$) such that $\mathcal{W}_k = \uparrow S_1^k \cup \ldots \cup \uparrow S_{l_k}^k$.*

(ii) For each $k \in N$, $S_1^k \cap \cdots \cap S_{l_k}^k \neq \emptyset$.

We denote by $\mathcal{G}$ the set of all normal command games (viewed as a subset of $2^{N \times 2^N}$).

We turn to the study of command functions ω, which are mappings from 2^N to 2^N. Clearly, the set of all such mappings has the same cardinality as the set of functions Ω, which is 2^{n2^n}. There exists an obvious bijection between $2^{N\times 2^N}$ and $(2^N)^{(2^N)}$, let us call it Ψ, defined by

$$\Psi(\Omega) = \omega, \quad \text{with } \omega(S) := \{k \in N \mid \Omega(k,S) = 1\}, \quad \forall S \subseteq N$$
$$\Psi^{-1}(\omega) = \Omega, \quad \text{with } \Omega(k,S) = 1 \text{ iff } k \in \omega(S). \tag{4}$$

Hence, ω and Ω are equivalent representations of a command game.

Given a function ω from 2^N to 2^N, what are the sufficient and necessary conditions so that $\Psi^{-1}(\omega)$ is a normal command game? The following proposition answers this question.

Proposition 6. *Let $\omega \in (2^N)^{(2^N)}$. Then ω corresponds to some normal command game, i.e., $\omega \in \Psi(\mathcal{G})$, if and only if the following conditions are satisfied:*

(i) $\omega(\emptyset) = \emptyset$, $\omega(N) = N$;

(ii) ω is isotone, i.e. it is monotone w.r.t. set inclusion;

(iii) If $S \cap S' = \emptyset$, then $\omega(S) \cap \omega(S') = \emptyset$.

Proof: Suppose ω corresponds to some normal command game. Then $\omega(\emptyset) = \emptyset$ follows from the fact that $\emptyset \notin \mathcal{W}_k$, $\forall k \in N$. On the other hand, $\omega(N) = N$ since $N \in \mathcal{W}_k$, $\forall k \in N$. Next, take $S \subseteq S' \subseteq N$. If $k \in \omega(S)$, then $k \in \omega(S')$ too due to the definition of $\mathcal{W}_k$, which proves that $\omega(S) \subseteq \omega(S')$. Lastly, if $k \in \omega(S) \cap \omega(S')$, then both S, S' belong to $\mathcal{W}_k$, and so they must have a nonempty intersection.

Conversely, assume that ω fulfills the three conditions, and consider $\Omega = \Psi^{-1}(\omega)$. Since $\omega(N) = N$, each $\mathcal{W}_k$ contains N, and thus is nonempty. Since $\omega(\emptyset) = \emptyset$, no $\mathcal{W}_k$ contains the emptyset. Take any $\mathcal{W}_k$, and consider $S \in \mathcal{W}_k$. Then any $S' \supseteq S$ belongs also to $\mathcal{W}_k$, since $S \subseteq S'$ implies $\omega(S) \subseteq \omega(S')$. This proves that $\mathcal{W}_k$ is an upset, hence it is a union of principal filters $\uparrow S_1^k, \dots, \uparrow S_{l_k}^k$. It remains to prove that there is no pair of disjoint sets in this family. Assuming $\mathcal{W}_k$ contains at least two subsets (otherwise the condition is void), take $S, S' \in \mathcal{W}_k$ such that $S \cap S' = \emptyset$. Then by (iii), $\omega(S) \cap \omega(S') = \emptyset$, which contradicts that fact that $S, S' \in \mathcal{W}_k$. ■

If $\omega \in \Psi(\mathcal{G})$, i.e., it satisfies the conditions of Proposition 6, then the notion of *kernel* of ω, denoted by $\mathcal{K}(\omega)$, is meaningful. It is the collection of minimal coalitions commanding at least one player:

$$\mathcal{K}(\omega) := \{S \in 2^N \mid \omega(S) \neq \emptyset, \text{ and } S' \subset S \Rightarrow \omega(S') = \emptyset\}.$$

Proposition 7. *Let $\omega \in \Psi(\mathcal{G})$. Then the unique command game $\{(N, \mathcal{W}_k), k \in N\}$ corresponding to ω is determined through its generating families $\{S_1^k, \dots, S_{l_k}^k\}$ of $\mathcal{W}_k$ as follows:*

$$\{S_1^k, \dots, S_{l_k}^k\} = \{S \in 2^N \mid \omega(S) \ni k \text{ and } S' \subset S \Rightarrow \omega(S') \not\ni k\}.$$

Proof: Clear from the previous development. Uniqueness comes from the fact that Ψ is one-to-one. ■

4.3 Relation between command games and influence functions

We turn to the relation between command games and influence functions. Since command functions and follower functions convey a similar meaning, the following definition is natural.

Definition 2. *Let B be an influence function and Ω be a command game. Then B and Ω are* equivalent *if $F_B = \omega$.*

Due to the previous results, equivalence between influence functions and command games is elucidated, and constitutes the main result of Section 4.

Theorem 2. *(i) Let B be an influence function. Then there exists a unique normal command game Ω equivalent to B if and only if $F_B(N) = N$. The generating families $\{S_1^k, \dots, S_{l_k}^k\}$, $k \in N$, of Ω are given by Proposition 7, taking $\omega := F_B$. The minimal boss sets and minimal approval sets are:*

$$\mathsf{Boss}_k^* = \{S_j^k \mid S_j^k \not\ni k, j = 1, \dots, l_k\}, \quad \mathsf{App}_k^* = \{S_j^k \setminus k \mid S_j^k \ni k, j = 1, \dots, l_k\}.$$

(ii) Let Ω be a normal command game. Then any influence function in $\Phi^{-1}(\omega)$ is equivalent to Ω, in particular the upper inverse $\overline{B}_\omega$ and the lower inverse $\underline{B}_\omega$. Moreover, the kernel of any influence function B in $\Phi^{-1}(\omega)$ is given by

$$\mathcal{K}(B) = \min\Big(\bigcup_{k \in N} \{S_1^k, \dots, S_{l_k}^k\}\Big) = \mathcal{K}(\omega)$$

where min(...) means that only minimal sets are selected from the collection.

In order to illustrate the concepts and results presented in this section, we recall two examples of command games mentioned in [18]. According to Definition 1, both are normal command games.

Example 5. Let us analyze the following command game:

$$N = \{1, 2, 3\}, \quad \mathcal{W}_1 = \{12, 13, 23, 123\}, \quad \mathcal{W}_2 = \{12, 23, 123\}, \quad \mathcal{W}_3 = \{23, 123\}.$$

The command function is:

$$\omega(1) = \omega(2) = \omega(3) = \emptyset, \quad \omega(12) = \{1, 2\}, \quad \omega(13) = \{1\}, \quad \omega(23) = N, \omega(N) = N$$
$$\mathcal{K}(\omega) = \{12, 13, 23\}.$$

We can apply Proposition 7 and Theorem 2 to this game, which gives

$$\{S_1^1, \dots, S_{l_1}^1\} = \{12, 13, 23\}, \quad \{S_1^2, \dots, S_{l_2}^2\} = \{12, 23\}, \quad \{S_1^3, \dots, S_{l_3}^3\} = \{23\}$$

$$\mathsf{Boss}_1^* = \mathsf{Boss}_1 = \{23\}, \quad \mathsf{Boss}_k^* = \mathsf{Boss}_k = \emptyset, \text{ for } k = 2, 3$$

$$\mathsf{App}_1^* = \mathsf{App}_1 = \{2, 3\}, \quad \mathsf{App}_2^* = \{1, 3\}, \quad \mathsf{App}_2 = \{1, 3, 13\} \quad \mathsf{App}_3^* = \{2\}, \quad \mathsf{App}_3 = \{2, 12\}.$$

It can be checked that the same result is obtained from the families of winning coalitions $\mathcal{W}_1$, $\mathcal{W}_2$, and $\mathcal{W}_3$.

The upper and lower inverses of ω are

S	$\emptyset$	1	2	3	12	13	23	N
$\overline{B}_\omega(S)$	$\emptyset$	$\emptyset$	23	3	N	N	N	N
$\underline{B}_\omega(S)$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	12	1	N	N

We have, of course, $F_{\underline{B}} = \omega$, $F_{\overline{B}} = \omega$, and

$$\mathcal{K}(\underline{B}) = \mathcal{K}(\overline{B}) = \{12, 13, 23\} = \min\Big(\bigcup_{k\in\{1,2,3\}} \{S^k_1, \dots, S^k_{l_k}\}\Big).$$

Example 6. Another example mentioned in [17] and later analyzed in [13] is the Confucian model of society, with

$$N = \{1, 2, 3, 4\}, \quad \mathcal{W}_1 = \{1234\}$$
$$\mathcal{W}_2 = \{1, 12, 13, 14, 123, 124, 134, 1234\}$$
$$\mathcal{W}_3 = \mathcal{W}_4 = \{2, 12, 23, 24, 123, 124, 234, 1234\}.$$

We have therefore

$$\omega(1) = \omega(13) = \omega(14) = \omega(134) = \{2\}, \quad \omega(2) = \omega(23) = \omega(24) = \omega(234) = \{3, 4\}$$
$$\omega(3) = \omega(4) = \omega(34) = \emptyset, \quad \omega(12) = \omega(123) = \omega(124) = \{2, 3, 4\}, \omega(N) = N$$
$$\mathcal{K}(\omega) = \{\{1\}, \{2\}\}.$$

Let us apply Proposition 7 to this game. We obtain

$$\{S^1_1, \dots, S^1_{l_1}\} = \{1234\}$$
$$\{S^2_1, \dots, S^2_{l_2}\} = \{1\}$$
$$\{S^3_1, \dots, S^3_{l_3}\} = \{S^4_1, \dots, S^4_{l_4}\} = \{2\}.$$

We can also apply Theorem 2 to this game:

$$\mathsf{Boss}^*_1 = \mathsf{Boss}_1 = \emptyset, \quad \mathsf{App}^*_1 = \mathsf{App}_1 = \{234\}$$
$$\mathsf{Boss}^*_2 = \{1\}, \quad \mathsf{Boss}_2 = \{1, 13, 14, 134\}$$
$$\mathsf{Boss}^*_3 = \mathsf{Boss}^*_4 = \{2\}, \quad \mathsf{Boss}_3 = \{2, 12, 24, 124\}, \quad \mathsf{Boss}_4 = \{2, 12, 23, 123\}$$
$$\mathsf{App}_k = \mathsf{App}^*_k = \emptyset \text{ for } k = 2, 3, 4.$$

The upper and lower inverses of ω are

S	$\emptyset$	1	2	3	4	12	13	14
$\overline{B}_\omega(S)$	$\emptyset$	12	134	1	1	N	12	12
$\underline{B}_\omega(S)$	$\emptyset$	2	34	$\emptyset$	$\emptyset$	234	2	2

S	23	24	34	123	124	134	234	N
$\overline{B}_\omega(S)$	134	134	1	N	N	12	134	N
$\underline{B}_\omega(S)$	34	34	$\emptyset$	234	234	2	34	N

We have, $F_{\underline{B}} = \omega$, $F_{\overline{B}} = \omega$, and

$$\mathcal{K}(\underline{B}) = \mathcal{K}(\overline{B}) = \{\{1\}, \{2\}\} = \min\Big(\bigcup_{k\in\{1,2,3,4\}} \{S^k_1, \dots, S^k_{l_k}\}\Big).$$

5 Summary of results and concluding remarks

We have tried in this paper to make clear the relationship between two different frameworks for the modeling of influence, namely influence functions and command games. We think useful to emphasize some results and points raised in the paper:

- The notion of *equivalence* between a command game and an influence function (see Definition 2) is the key notion permitting to compare the two frameworks, and this notion is naturally dictated by the definitions of follower and command functions. Moreover, the notion of equivalence permits also to clarify the operational meaning of command games. Indeed, in the framework of Hu and Shapley, it is not clearly stated, once the winning coalitions, boss sets and approval sets are fixed, what finally the players will decide in a given voting situation. The link we propose through the follower function permits to know all possible decision vectors from a given inclination vector. Specifically, given a command game Ω, we compute $\omega = \Psi(\Omega)$, then considering ω as a follower function, we compute $\Phi^{-1}(\omega)$, which is the set of all possible influence functions equivalent to Ω. From a given inclination vector i, the set of all possible decision vectors under the command game Ω is $\{Bi \mid B \in \Phi^{-1}(\Psi(\Omega))\}$.
- The framework of influence functions is more general than the framework of command games in two aspects. Theorem 2 shows clearly that, firstly, there are influence functions not representable by a normal command game (these are all B's such that $F_B(N) \neq N$), and secondly, to each normal command game, it corresponds in general several influence functions which are equivalent to the command game.
- On the other hand, the framework of command games brings an interesting interpretation of the framework of influence functions. The generality of influence functions has to be paid by a relative opacity of its meaning. Given an influence function B, it is hard to directly guess what are the influential players, and what is the exact mechanism of influence implemented by B. Provided an equivalent command game exists, Theorem 2 (i) brings a nice interpretation of an influence function through (minimal) boss sets and approval sets. One should note also that boss and approval sets are closely linked to the notion of kernel (of an influence function or a command function).

Figure 2 tries to make clear the above points.

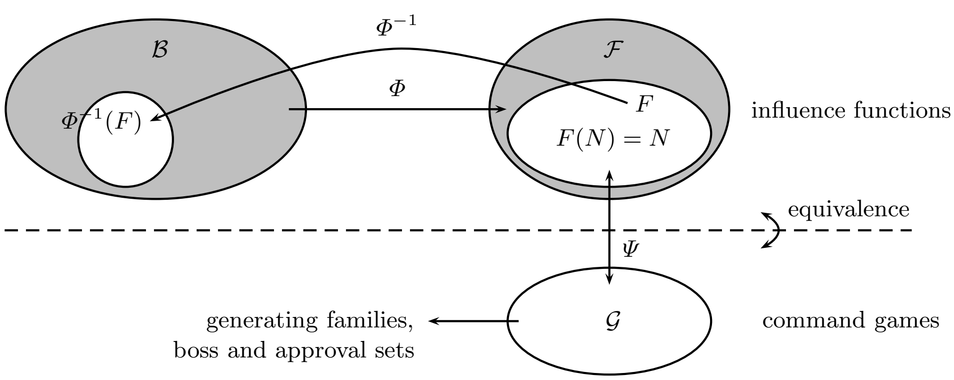

Fig. 2. Relations between influence functions and command games

The results presented in this paper establish a basis on which further research can be undertaken. For instance, a generalized model of influence, in which each player has an ordered set of possible actions, should be investigated. The main aim of this research line is to follow some works on abstention that have already been presented in the literature on voting (see, e.g. [6–8, 24]), and on multi-choice games [16]. Related models are games with r alternatives, where the alternatives are not ordered; see [1–4]. Also in [9–11] the authors consider voting systems with several levels of approval in the input and output, where those levels are qualitatively ordered. They introduce (j, k) simple games, in which each voter expresses one of j possible levels of input support, and the output consists of one of k possible levels of collective support. Standard simple games are therefore $(2, 2)$ simple games, and $(3, 2)$ simple games allow each voter a middle option, which may be interpreted as abstention.

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A Elementary notions of lattices and posets

We give here the necessary material on posets and lattices which are essentially used for Theorem 1. For further reference, we recommend [5]. In all the section, sets are finite.

A *lattice* $(L, \leq)$ is a partially ordered set such that for any $x, y \in L$ their least upper bound, denoted by $x \vee y$, and greatest lower bound $x \wedge y$ always exist. When L is finite, there always exist a greatest element and a least element in L, called the *top* and the *bottom* element, and denoted by $\top, \perp$.

For example, the poset $(2^N, \subseteq)$ where N is a finite set, is an example of lattice, where $\vee, \wedge$ are respectively $\cup, \cap$. Top and bottom elements are N and $\emptyset$ respectively.

A lattice is *autodual* if reversing the order relation, the same lattice is obtained (up to an isomorphism). $(2^N, \subseteq)$ is autodual, since replacing any subset S by $N \setminus S$, we get the same structure.

Consider a lattice $(L, \leq)$ and $L' \subseteq L$. Then $(L', \leq)$ is a *sublattice* of L if it is a lattice, and the supremum and infimum of $(L', \leq)$ coincide with those of $(L, \leq)$.

Let $(P, \leq)$ be any poset. $Q \subseteq P$ is a *downset* of P if $x \in Q$ and $y \leq x$ imply $y \in Q$. The set of all downsets of P is denoted by $\mathcal{O}(P)$.

For any two elements $x, y \in L$, x *is covered by* y or y *covers* x (denoted by $x \prec y$ or $y \succ x$) if $x < y$ and there is no z such that $x < z < y$. A sequence of elements such that $x \leq y_1 \leq y_2 \leq \cdots \leq z$ is called a *chain* from x to z. If in addition $x \prec y_1 \prec y_2 \cdots \prec z$, the chain is *maximal*. An *antichain* is a set of elements such that there is no pair of comparable elements in it. In $(2^N, \subseteq)$, any collection of subsets of the same cardinality is an antichain.

An element $j \in L$ is *join-irreducible* if it is not the bottom element and it cannot be expressed as a supremum of other elements, or equivalently, if it covers only one element. The set of all join-irreducible elements is denoted by $\mathcal{J}(L)$.

The lattice is said to be *distributive* if $\vee, \wedge$ obey distributivity. Every sublattice of a distributive lattice is distributive. When the lattice is distributive, any element $x \in L$ can be expressed in a unique way as an irredundant supremum of join-irreducible elements. This means that if L is distributive, it suffices to know only $\mathcal{J}(L)$ to reconstruct all the lattice. More precisely, L is isomorphic to $\mathcal{O}(\mathcal{J}(L))$, the set of all downsets of $\mathcal{J}(L)$ (Birkhoff's theorem).

The *height function* h on L gives the length of a longest chain from $\perp$ to any element in L. The *height* of the lattice is $h(\top)$. A lattice is *ranked* if $x \succ y$ implies $h(x) = h(y)+1$. If a lattice is distributive, then it is ranked and the length of any maximal chain from $\perp$ to $\top$ is $|\mathcal{J}(L)|$.

A lattice is *Boolean* if it is isomorphic to some lattice of subsets $(2^N, \subseteq)$. Every Boolean lattice is distributive.

Consider lattices $(L_1, \leq_1), \ldots, (L_n, \leq_n)$. The *product lattice* of these lattices is the lattice $(L, \leq)$, with $L := L_1 \times \cdots \times L_n$ (also denoted by $\prod_{i=1}^{n} L_i$), and $\leq$ is the product order defined by, for any $x := (x_1, \ldots, x_n)$, $y := (y_1 \ldots, y_n)$ in L,

$$x \leq y \Leftrightarrow x_i \leq_i y_i, i = 1, \ldots, n.$$

The product of a family of Boolean lattices is itself a Boolean lattice.

A function $\tau : (L, \leq) \to (L, \leq)$ is a *dual closure operator* if $\tau(\top) = \top$, $x \geq \tau(x)$ for all $x \in L$, τ is isotone, and $\tau(\tau(x)) = \tau(x)$.

B Proof of Proposition 3

We express $F_B(S)$ and see under which conditions the equality $F_B(S) = F(S)$ holds for all $S \subseteq N$. Let us remark that the equality always holds for $S = \emptyset$.

Using (1), the definition of B, and Figure 1 we have, for any $S \subseteq N$, $S \neq \emptyset$,

$$\begin{aligned}
F_B(S) &= \bigcap_{S' \supseteq S} \big(F(S') \cup T_{S'}\big) \cap \bigcap_{S' \subseteq N \setminus S} \big(\overline{F(S')} \cap \overline{T_{S'}}\big) \\
&= \bigcap_{S' \supseteq S} \big(F(S') \cup T_{S'}\big) \cap \bigcap_{S' \supseteq S} \big(\overline{F(\overline{S'})} \cap \overline{T_{\overline{S'}}}\big) \\
&= \bigcap_{S' \supseteq S} \Big[\big(F(S') \cup T_{S'}\big) \cap \big(\overline{F(\overline{S'})} \cap \overline{T_{\overline{S'}}}\big)\Big] \\
&= \bigcap_{S' \supseteq S} \Big[\big(\underbrace{F(S') \cap \overline{F(\overline{S'})} \cap \overline{T_{\overline{S'}}}}_{F(S')}\big) \cup \big(\underbrace{T_{S'} \cap \overline{F(\overline{S'})} \cap \overline{T_{\overline{S'}}}}_{T_{S'} \setminus T_{\overline{S'}}}\big)\Big] \\
&= \bigcap_{S' \supseteq S} \big(F(S') \cup (T_{S'} \setminus T_{\overline{S'}})\big).
\end{aligned}$$

If $S = N$, then we obtain $F_B(N) = F(N) \cup (T_N \setminus T_\emptyset)$, which implies $T_N \setminus T_\emptyset = \emptyset$. This is condition (i) for $S = N$ (in its long version, see Remark 3). Since condition (ii) is void for $S = N$, we have proved the result for $S = N$.

Consider $S \neq \emptyset, N$. Applying distributivity of $\cap$ and $\cup$, we obtain:

$$F_B(S) = \bigcup_{\mathcal{S} \subseteq [S,N]} \big(\bigcap_{S' \in \mathcal{S}} F(S') \cap \bigcap_{S' \in [S,N] \setminus \mathcal{S}} T_{S'} \setminus T_{\overline{S'}}\big).$$

Let us process first the simple cases where $\mathcal{S} = \varnothing$ and $\mathcal{S} = [S, N]$. In the first case, we get only the term $\bigcap_{S' \supseteq S} T_{S'} \setminus T_{\overline{S'}}$. In the second case, it remains only $\bigcap_{S' \supseteq S} F(S')$, which is equal to $F(S)$ by isotonicity of F.

We suppose now that $\mathcal{S} \neq \varnothing, [S, N]$. Suppose $\mathcal{S} \ni S$. Then by isotonicity of F we get $\bigcap_{S' \in \mathcal{S}} F(S') = F(S)$. Moreover, by Figure 1 and isotonicity of F again, we obtain that $F(S) \cap \bigcap_{S' \in [S,N] \setminus \mathcal{S}} T_{S'} \setminus T_{\overline{S'}} = \emptyset$.

Suppose on the contrary that $\varnothing \neq \mathcal{S} \in]S, N]$. By isotonicity of F, if the family $\mathcal{S}$ has a single minimal element, say S_1 (i.e., $\mathcal{S} \subseteq [S_1, N]$ and $S_1 \in \mathcal{S}$), then $\bigcap_{S' \in \mathcal{S}} F(S') = F(S_1)$. Similarly, if the family $\mathcal{S}$ has two minimal elements, say S_1, S_2, then $\bigcap_{S' \in \mathcal{S}} F(S') = F(S_1) \cap F(S_2)$, and so on. Let us consider all families $\mathcal{S}$ having as single minimal element S_1, and consider $\bigcap_{S' \in [S,N] \setminus \mathcal{S}} T_{S'} \setminus T_{\overline{S'}}$. The largest set will be obtained for the smallest family $[S, N] \setminus \mathcal{S}$, hence the largest $\mathcal{S}$, which is $[S_1, N]$. Hence

$$\bigcup_{\substack{\mathcal{S} \subseteq]S,N] \\ \text{minimal element}=S_1}} \big((F(S_1) \cap \bigcap_{S' \in [S,N] \setminus \mathcal{S}} T_{S'} \setminus T_{\overline{S'}}\big) = F(S_1) \cap \bigcap_{\substack{S' \supseteq S \\ S' \not\supseteq S_1}} T_{S'} \setminus T_{\overline{S'}}.$$

The reasoning easily extends to any number of minimal elements:

$$\bigcup_{\substack{\mathcal{S} \subseteq]S,N] \\ \text{minimal elements}=S_1,S_2}} \big((F(S_1) \cap F(S_2) \cap \bigcap_{S' \in [S,N] \setminus \mathcal{S}} T_{S'} \setminus T_{\overline{S'}}\big) = F(S_1) \cap F(S_2) \cap \bigcap_{\substack{S' \supseteq S \\ S' \not\supseteq S_1, S_2}} T_{S'} \setminus T_{\overline{S'}}$$

and so on. So we obtain in summary:

$$F_B(S) = \underbrace{F(S)}_{\mathcal{S}=[S,N]} \cup \underbrace{\Big(\bigcap_{S' \supseteq S} T_{S'} \setminus T_{\overline{S'}}\Big)}_{\mathcal{S}=\emptyset} \cup \bigcup_{\substack{\text{all antichains } \mathcal{C} \text{ in }]S,N] \\ \bigcap_{S'\in\mathcal{C}} F(S') \neq F(S)}} \Big(\bigcap_{S' \in \mathcal{C}} F(S') \cap \bigcap_{\substack{S'' \supseteq S \\ S'' \not\supseteq S', \forall S' \in \mathcal{C}}} T_{S''} \setminus T_{\overline{S''}} \Big).$$

To get this final expression, observe that the set of all possible configurations of minimal elements of $\mathcal{S}$ coincides with the set of antichains of $]S,N]$. Finally, observe that for any antichain $\mathcal{C}$ in $]S,N]$, we have $\bigcap_{S'\in\mathcal{C}} F(S') \supseteq F(S)$. If equality occurs, then the intersection with any $T_{S'} \setminus T_{\overline{S'}}$, $S' \in [S,N]$ is empty, by Figure 1 and isotonicity of F.

Then clearly, $F_B(S) = F(S)$ if and only if each term in parenthesis is the empty set.

C Proof of Lemma 1

Since the only change between B and B' is the addition of k to S, the only change between F_B and $F_{B'}$ concerns k. Hence, we have to consider, for any $S' \subseteq N$, the occurrence of two situations: either (i) k is a follower of S' for B but no more for B', or (ii) k is not a follower of S' for B but it becomes for B'.

First observe that the case $S' = S$ is done, hence we can discard it from the analysis. Indeed, $k \notin F_{B'}(S)$ by hypothesis, and $k \in F_B(S)$ is impossible since $k \notin B(S)$. Hence $F_B(S) = F_{B'}(S)$.

(i) We consider that $k \in F_B(S')$ and $k \notin F_{B'}(S')$. Then for all $S'' \supseteq S'$, we have $k \in B(S'')$, and for all $S'' \subseteq N \setminus S'$, $k \notin B(S'')$.

Suppose that $S' \supset S$ or S and S' are incomparable with a nonempty intersection. Then S is neither a superset of S' nor a subset of $N \setminus S'$, which means that B and B' are identical on $(\uparrow S') \cup (\downarrow \overline{S'})$. Then $k \in F_{B'}(S')$, a contradiction.

Suppose that $S' \subset S$. Since S is a superset of S', by hypothesis we have $k \in B(S)$, which contradicts the definition of B.

The remaining case is $S \cap S' = \emptyset$. Then any superset of $\overline{S}$ is a superset of S', and any subset of $N \setminus \overline{S} = S$ is a subset of $N \setminus S'$, which by hypothesis implies that $k \in F_B(\overline{S})$. But this contradicts the assumption.

(ii) We consider that $k \notin F_B(S')$ and $k \in F_{B'}(S')$. Then for all $S'' \supseteq S'$, we have $k \in B'(S'')$, and for all $S'' \subseteq N \setminus S'$, $k \notin B'(S'')$.

Suppose that $S' \supset S$ or S and S' are incomparable with a nonempty intersection. As said above, B and B' are identical on $(\uparrow S') \cup (\downarrow \overline{S'})$. Then $k \in F_B(S')$, a contradiction.

Suppose that $S' \subset S$. Then any superset of S is a superset of S', and any subset of $N \setminus S$ is a subset of $N \setminus S'$, which by hypothesis implies that $k \in F_{B'}(S)$, a contradiction with the assumption.

Finally, suppose that $S \cap S' = \emptyset$. Since S is a subset of $N \setminus S'$, we have $k \notin B'(S)$, which contradicts the definition of B'.

D Proof of Theorem 1

(i) is already known from Prop. 1.

Let us prove first (iii). Take $B := (T_\emptyset, \dots, T_S, \dots, T_N)$ of $\Phi^{-1}(F)$. Then the two conditions of Prop. 3 hold for all $S \in 2^N$. We have to prove that they still hold for B', which amounts to replace in these two conditions $T_{S'} \setminus T_{\overline{S'}}$ by $(D_{\overline{S'}} \setminus T_{\overline{S'}}) \setminus (D_{S'} \setminus T_{S'})$.

But since $D_{S'} = \overline{F(S')} \setminus F(\overline{S'}) = D_{\overline{S'}}$, and by Figure 1, $(D_{\overline{S'}} \setminus T_{\overline{S'}}) \setminus (D_{S'} \setminus T_{S'}) = T_{S'} \setminus T_{\overline{S'}}$, which proves the result.

Let us prove (ii). Since $\Phi^{-1}(F)$ is a subset of a product of set lattices, the operations $\vee, \wedge$ defined above are clearly supremum and infimum. We just have to prove that $B \vee B'$ and $B \wedge B'$ belong to $\Phi^{-1}(F)$ whenever B, B' belong to $\Phi^{-1}(F)$, to prove that Φ^{-1} is a lattice. Due to (iii), we only have to prove it for, e.g., the infimum.

Consider $B := (T_\emptyset, \dots, T_N)$ and $B' = (T'_\emptyset, \dots, T'_N)$ in $\Phi^{-1}(F)$. Then $B \wedge B' = (T_\emptyset \cap T'_\emptyset, \dots, T_N \cap T'_N)$. Using Proposition 3, we have to prove that the two conditions there are satisfied. The first one reads

$$K := \bigcap_{S' \supseteq S} (T_{S'} \cap T'_{S'}) \setminus (T_{\overline{S'}} \cap T'_{\overline{S'}}) = \emptyset, \quad \forall S \subseteq N, S \neq \emptyset.$$

From the general relation

$$(A \cap A') \setminus (B \cap B') = \big((A \setminus B) \cap A'\big) \cup \big((A' \setminus B') \cap A\big)$$

we get

$$K = \bigcap_{S' \supseteq S} \Big[\big((T_{S'} \setminus T_{\overline{S'}}) \cap T'_{S'}\big) \cup \big((T'_{S'} \setminus T'_{\overline{S'}}) \cap T_{S'}\big)\Big].$$

Applying distributivity we get

$$K = \bigcup_{\mathcal{S} \subseteq [S,N]} \Big[\bigcap_{S' \in \mathcal{S}} (T_{S'} \setminus T_{\overline{S'}}) \cap T'_{S'} \cap \bigcap_{S' \in [S,N] \setminus \mathcal{S}} (T'_{S'} \setminus T'_{\overline{S'}}) \cap T_{S'}\Big] =: \bigcup_{\mathcal{S} \subseteq [S,N]} K_{\mathcal{S}}.$$

Taking $\mathcal{S} = [S,N]$ or $\varnothing$, $K_{\mathcal{S}} = \emptyset$, because by Proposition 3 we have $\bigcap_{S' \in [S,N]} T_{S'} \setminus T_{\overline{S'}} = \emptyset$, and the same for B'. We consider then $K_{\mathcal{S}}$ with $\varnothing \neq \mathcal{S} \subset [S,N]$. Suppose that $K_{\mathcal{S}} \neq \emptyset$, and take any $x \in K_{\mathcal{S}}$. Then we deduce that

(a) $x \in T_{S'}$ and $x \in T'_{S'}$ for all $S' \in [S,N]$;
(b) $x \notin T_{\overline{S'}}$ for all $S' \in \mathcal{S}$, and $x \notin T'_{\overline{S'}}$ for all $S' \in [S,N] \setminus \mathcal{S}$.

From (a), we easily deduce that for all $S' \supseteq S$ and all $S' \subseteq N \setminus S$, $x \notin F(S')$. Indeed, from (a), we know that $x \in D_{S'}$, and so $x \notin F(S')$ for all $S' \in [S,N]$. Next, for any $S' \subseteq N \setminus S$, we have $\overline{S'} \in [S,N]$. Then $x \in D_{\overline{S'}} = D_{S'}$, which proves again that $x \notin F(S')$.

From this we deduce that in particular $x \notin F(N)$. Let us prove that x necessarily belongs to either $F_B(N)$ or $F_{B'}(N)$, which causes $F \neq F_B$ or $F \neq F_{B'}$, a contradiction with the hypothesis. This amounts to prove that x is a follower of N for B or B'. We know by (a) that $x \in T_N$ and $x \in T'_N$, which proves that $x \in B(N)$ and $B'(N)$. By (b), we deduce that $x \notin T_\emptyset$ (if $N \in \mathcal{S}$) or $x \notin T'_\emptyset$ (if $N \notin \mathcal{S}$). Since $F(\emptyset) = \emptyset$, we deduce that $x \notin B(\emptyset)$ or $x \notin B'(\emptyset)$.

We turn to the second condition, which reads

$$K := \bigcap_{S' \in \mathcal{C}} F(S') \cap \bigcap_{S' \in \mathcal{S}_{\mathcal{C}}} (T_{S'} \cap T'_{S'}) \setminus (T_{\overline{S'}} \cap T'_{\overline{S'}}) = \emptyset,$$

for all antichain $\mathcal{C} \in]S,N]$, for all $S \subset N$, and $\mathcal{S}_{\mathcal{C}} := \{S'' \supseteq S, S'' \not\supseteq S', \forall S' \in \mathcal{C}\}$. Proceeding as above, we get after some manipulation

$$K = \bigcup_{\mathcal{S} \subseteq \mathcal{S}_{\mathcal{C}}} \Big[\bigcap_{S' \in \mathcal{C}} F(S') \cap \bigcap_{S' \in \mathcal{S}} \big((T_{S'} \setminus T_{\overline{S'}}) \cap T'_{S'}\big) \cap \bigcap_{S' \in \mathcal{S}_{\mathcal{C}} \setminus \mathcal{S}} \big((T'_{S'} \setminus T'_{\overline{S'}}) \cap T_{S'}\big) \Big] =: \bigcup_{\mathcal{S} \subseteq \mathcal{S}_{\mathcal{C}}} K_{\mathcal{S}}.$$

We have to prove that $K_{\mathcal{S}} = \emptyset$, $\forall \mathcal{S} \subseteq \mathcal{S}_{\mathcal{C}}$. As above, taking $\mathcal{S} = \mathcal{S}_{\mathcal{C}}$ or $\varnothing$ leads to $K_{\mathcal{S}} = \emptyset$ by Proposition 3. We consider then $\varnothing \neq \mathcal{S} \subset \mathcal{S}_{\mathcal{C}}$, and we assume that $K_{\mathcal{S}} \neq \emptyset$, and consider $x \in K_{\mathcal{S}}$. This implies that x belongs to each term of the intersections in $K_{\mathcal{S}}$, hence

(c) $x \in T_{S'}$ and $x \in T'_{S'}$ for all $S' \in \mathcal{S}_{\mathcal{C}}$;
(d) $x \notin T_{\overline{S'}}$ for all $S' \in \mathcal{S}$, and $x \notin T'_{\overline{S'}}$ for all $S' \in \mathcal{S}_{\mathcal{C}} \setminus \mathcal{S}$;
(e) $x \in F(S')$, $\forall S' \in \mathcal{C}$.

Proceeding as above again, we deduce from (c) that

$$x \notin F(S') \text{ for all } S' \in \mathcal{S}_{\mathcal{C}}, \text{ and for all } S' \subseteq N \setminus S \text{ such that } \overline{S'} \in \mathcal{S}_{\mathcal{C}}. \tag{5}$$

Consider S_0, any maximal element of $\mathcal{S}_{\mathcal{C}}$. An important fact is to notice that

$$\forall S' \supset S_0, \quad \exists S'' \in \mathcal{C} \text{ such that } S' \supseteq S''. \tag{6}$$

Indeed, $S' \supset S_0$ implies $S' \notin \mathcal{S}_{\mathcal{C}}$. On the other hand, $S' \subseteq N$ and $S' \supset S_0 \supseteq S$ implies $S' \in [S,N]$. So by definition of $\mathcal{S}_{\mathcal{C}}$, $S' \supseteq S''$ for some $S'' \in \mathcal{C}$.

Let us prove that $x \in F_B(S_0)$ or $F_{B'}(S_0)$. Since $x \notin F(S_0)$ by (5), this suffices to give a contradiction, hence $K_{\mathcal{S}} = \emptyset$ for all $\mathcal{S}$, which proves that the second condition is fulfilled. For this, we have to prove:

- $\forall S' \supseteq S_0$, $B(S') \ni x$ (or the same with B'). This is true for $S' = S_0$ by (c). By (6), $S' \supseteq S''$ for some $S'' \in \mathcal{C}$. Since $x \in F(S'')$ by assumption, $x \in F(S')$ too by isotonicity of F. Hence, $x \in B(S')$, and also $x \in B'(S')$.
- $\forall S' \subseteq N \setminus S_0$, $B(S') \not\ni x$ (or the same condition with B'). From (5) we know that $x \notin F(N \setminus S_0)$, hence $x \notin F(S')$ for all $S' \subseteq N \setminus S_0$ by isotonicity of F. So it remains to show that $x \notin T_{S'}$ for all $S' \subseteq N \setminus S_0$ (or the same with $T'_{S'}$). By (d), we have $x \notin T_{\overline{S_0}}$ or $x \notin T'_{\overline{S_0}}$ (depending whether $S_0 \in \mathcal{S}$ or not). Let us assume the case $x \notin T_{\overline{S_0}}$. By (e), $x \in F(S'')$ for all $S'' \in \mathcal{C}$, and $F = F_B = F_{B'}$. This implies that for all $S''' \subseteq N \setminus S''$, $S'' \in \mathcal{C}$, $x \notin T_{S'''}$. It remains to prove that any $S' \subset N \setminus S_0$ is necessarily a subset of $N \setminus S''$ for some $S'' \in \mathcal{C}$. But this is equivalent to prove that any $S' \supset S_0$ is a superset of some $S'' \in \mathcal{C}$, which is exactly (6).

The proof that $B \wedge B'$ belongs to $\Phi^{-1}(F)$ is complete. Now, since these infimum and supremum are those of $\prod_{S \subseteq N} 2^{D_S}$, $\Phi^{-1}(F)$ is a sublattice of it.

(iv) and (v). First, since $\Phi^{-1}(F)$ is a sublattice of a distributive lattice, it is distributive. Hence, it is ranked and the length of any maximal chain from bottom to top is the number of join-irreducible elements. Second, the height of the lattice is at most $\sum_{S \subseteq N} |D_S|$ since one adds at least one new element k of some D_S at each step, hence this is the maximal number of join-irreducible elements.

We examine now what are the join-irreducible elements. Let us take $S \subseteq N$ such that $D_S \neq \emptyset$, and any $k \in D_S$. Evidently, $(k_S\emptyset)$ covers only one element, the bottom of the lattice. Hence it is a join-irreducible element provided it belongs to $\Phi^{-1}(F)$. Applying Proposition 4, this amounts to show that at least one of the following conditions is false:

(i) $\forall S' \supset S$, $k \in F(S')$
(ii) $\forall S' \subseteq N \setminus S$, $k \notin F(S')$.

If one of the conditions fails, then $(k_S\emptyset)$ is a join-irreducible element. Suppose then that both conditions hold. Then $(k_S\emptyset) \notin \Phi^{-1}(F)$ because k becomes a follower of S. It suffices to add k at some position S' with $S' \subseteq N \setminus S$, to prevent k from becoming a follower of S. Observe that $S' = N \setminus S$ is always a solution, because $D_S = D_{N\setminus S}$, and so $k \in D_{N\setminus S}$. Moreover, it is the only solution, since for any $S' \subset N \setminus S$, $k \notin D_{S'}$. Indeed, if $k \in D_{S'}$, then $k \in D_{\overline{S'}}$, which means that $k \notin F(\overline{S'})$. But $\overline{S'}$ is a proper superset of S, so this contradicts assumption (i). Hence $(k_S k_{\overline{S}}\emptyset)$ is an element of $\Phi^{-1}(F)$. Lastly, we have to show that $(k_S k_{\overline{S}}\emptyset)$ covers only one element. Since by assumption $(k_S\emptyset)$ is not an element of $\Phi^{-1}(F)$, it can only cover, either $(k_{\overline{S}}\emptyset)$ or, if this element does not belong to $\Phi^{-1}(F)$, the bottom element. In both cases we are done, but observe that only the first case can occur. Indeed, by hypothesis, condition (ii) holds, which makes fail condition (i) written for $\overline{S}$ instead of S.

Since doing so for all $S \subseteq N$ and all $k \in D_S$ we have found $\sum_{S\subseteq N} |D_S|$ join-irreducible elements, there cannot be more.